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# NON-STATIONARY PROBLEM OF A PLANE HYDRAULIC FAULT CRACK IN A FLUID-SATURATED STRATUM*

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The problem of a vertical hydraulic fault crack /1/ in a fluid-saturated stratum wedged out by a viscous filtering fluid flow is considered. It is assumed that the state of stress and strain of the stratum is described by a system of Biot equations /2/. A system of elastic constant notation, proposed in /3/, is used.

The non-stationary problem of a vertical hydraulic fault crack in a fluid-saturated stratum in one special case of representing the general solution of the consolidation theory equations reduces to the solution of an equation of piezoconductivity type with a source and a formula connecting displacement of the crack edges with the fault fluid pressure and the fluid leakage velocity through the crack walls. In the case of a fixed "ideal" crack, along which the pressure is constant, the problem of a hydraulic fault reduces to solving a one-dimensional singular integral equation for the Laplace transform. Asymptotic forms of the solution of this equation are found for long and short times. Representation of the general solution of the consolidation theory equations in the Papkovich-Neuber form was obtained to solve consolidation theory problem /4, 5/. Compressibility effects of the interstitial fluid /6/ were taken into account in the development of this method. A representation of the general solution of the consolidation theory equations in terms of two analytic functions of a complex variable /3/ was obtained in another approach to the solution of plane problems. Application of consolidation theory to the investigation of stationary problems of a hydraulic fault of a fluid-saturated stratum was started in /7, 8/.

*1. Formulation of the problem.* Let a plane crack in an infinite porous fluid-saturated space in a homogeneous compressive stress field $\sigma_0$ be maintained in an open state by fluid heated within the crack, which can filter through its wall into a porous medium while moving along the crack. It is assumed that the borehole radius $r_0$ can be less than the crack length $L_0$ and, consequently, effects associated with the presence of the borehole can be neglected.

In particular, this crack theory problem occurs in connection with the problem of a hydraulic fault in an oil-bearing stratum /1/.

A coupled theory of consolidation /3/ $(i, j, k = 1, 2, 3,$ and summation is over repeated subscripts) is used to describe the strain of a fluid-saturated porous medium and the filtration of the interstitial fluid therein:

$$d\sigma_{ij}/\partial x_j = 0, \quad \sigma_{ij} = \sigma_{ji} \tag{1.1}$$

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$$2G\varepsilon_{ij} = \sigma_{ij} - \left(\frac{\nu}{1+\nu}\sigma_{kk} - 2\eta\frac{1-\nu}{1+\nu}P\right)\delta_{ij}; \quad \eta = \frac{3(\nu_u - \nu)}{2B(1-\nu)(1+\nu_u)} \tag{1.2}$$

$$\frac{\partial}{\partial t}m + \frac{\partial}{\partial x_i}q_i = 0, \quad m - m_0 = \frac{\rho_0}{G}\frac{(1-\nu)}{(1+\nu)}\eta\left(\sigma_{kk} + \frac{3}{B}P\right) \tag{1.3}$$

$$q_i = \rho_0 u_i = -\rho_0\frac{k}{\mu}\frac{\partial}{\partial x_i}P \tag{1.4}$$

Here $\sigma_{ij}$ is the total stress tensor, $P$ is the porous pressure, $B$ is the Scampton parameter, $\nu$ is Poisson's ratio, $\nu_u$ is Poisson's ratio corresponding to conditions when the fluid cannot leave the medium /3/ $G$ is the shear modulus, $\varepsilon_{ij}$ is the strain tensor, $m$ is the mass of interstitial fluid per unit volume, $\rho_0, m_0$ are the density and mass of interstitial fluid per unit volume in the undeformed state, $k$ is a permeability factor, $\mu$ is the fluid viscosity, and $u_i$, $q_i$ are the rate of filtration and the mass flow rate of the fluid being filtered in the $i$-th direction.

For the plane state of strain $(\varepsilon_{33} = 0)$, which will also be examined below, Hooke's law for a fluid-saturated porous medium (1.2) takes the form $(\alpha, \beta, \gamma = 1, 2)$

$$2G\varepsilon_{\alpha\beta} = \sigma_{\alpha\beta} - \nu(\sigma_{\gamma\gamma} - 2(1-\nu)\eta P)\delta_{\alpha\beta} \tag{1.5}$$

The axis $x_2 = 0$ is selected to be along the crack with origin $(x_1 = 0, x_2 = 0)$ at the centre of the crack.

The motion of the heated fluid along the crack is described by the equation of continuity and Poiseuille's law

$$\frac{\partial}{\partial t}w + \frac{\partial}{\partial x_1}(wu) = -2v, \quad u = -\frac{w^2}{12\mu}\frac{\partial}{\partial x_1}P_c \tag{1.6}$$

Here $w$ is the opening of the crack edges, $P_c$ is the pressure of the fault fluid heated in the crack, $u$ is the velocity of fluid motion, and $v$ is the rate of fault fluid leakage through the crack wall.

The boundary conditions

$$P_c(x_1, t) = P(x_1, x_2 = 0, t) \tag{1.7}$$
$$v(x_1, t) = -\frac{k}{\mu}\frac{\partial}{\partial x_2}P(x_1, x_2 = 0, t)$$

are imposed on the crack edges.

**2. *Method of solution.*** The system (1.1), (1.3)-(1.5) for the plane state of strain was reduced to a problem of determining two analytic functions of a complex variable by methods of the theory of singular integral equations /3/. Unlike this approach, we obtain a generalization of the Kolosov formulas in the special case needed for the theory of a hydraulic fault crack in terms of an Airy function.

Satisfying the equlibrium Eqs.(1.1) for the plane problem identically $(\alpha, \beta = 1, 2)$, we introduce the Airy function

$$\sigma_{\alpha\beta} = (-1)^{\alpha+\beta}\partial^2 F/\partial x_\alpha \partial x_\beta \tag{2.1}$$

From the condition of strain compatibility

$$\partial^2\varepsilon_{11}/\partial x_2^2 + \partial^2\varepsilon_{22}/\partial x_1^2 = 2\partial^2\varepsilon_{12}/\partial x_1\partial x_2$$

and the Hooke's law (1.5), taking (2.1) into account, we obtain

$$\Delta^2 F = -2\eta\Delta P \quad (\Delta = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2) \tag{2.2}$$

We apply the theory of functions of a complex variable by using the following method /9/ to solve (2.2). We consider the independent variables $x_1$, $x_2$ and the functions $F$, $P$ to be complex variables. We convert them to new variables $z = x_1 + ix_2$, $\bar{z} = x_1 - ix_2$ which are independent in this case. On returning to the original variables, when $x_1, x_2$ are real, $z$ and $\bar{z}$ become conjugate values of one complex variable.

The representation of (2.2) in the variables $z$ and $\bar{z}$ has the form

$$\partial^4 F/\partial z^2\partial\bar{z}^2 = -{}^1/_2\eta\partial^2 P/\partial z\partial\bar{z} \tag{2.3}$$

Integrating (2.3) and taking into account that the Airy function $F$ should be real on changing to the real variables $x_1, x_2$, we obtain

$$F(z,\bar{z}) = -\frac{1}{2}\eta\int_{z_0}^{z}\int_{\bar{z}_0}^{\bar{z}} P\,d\zeta\,d\bar{\zeta} + \bar{z}\varphi(z) + z\bar{\varphi}(\bar{z}) + \chi(z) + \bar{\chi}(\bar{z})] \tag{2.4}$$

$\varphi(z)$, $\chi(z)$ are analytic functions and $z_0, \bar{z}_0$ are certain constants.
Substituting (2.4) into (2.1), we obtain after some reduction

$$\sigma_{11} + \sigma_{22} + 2\eta P = 2(\varphi'(z) + \bar{\varphi}'(\bar{z})) \tag{2.5}$$

$$\sigma_{11} - \sigma_{22} + 2i\sigma_{12} - 2\eta\frac{\partial}{\partial\bar{z}}\int_{z_0}^{z} P\,d\zeta = -2[z\bar{\varphi}''(\bar{z}) + \bar{\psi}''(\bar{z})] \quad (\psi = \chi')$$

We obtain from Hooke's law (1.5) and the definition of the strain tensor in terms of the displacement vector components

$$\sigma_{11} - \sigma_{22} + 2i\sigma_{12} = 4G\partial D/\partial\bar{z}, \quad D = w_1 + iw_2 \tag{2.6}$$

($w_i$ is the $i$-th component of the displacement vector $\mathbf{w}$).
Substituting (2.5) into (2.6) and integrating with respect to $\bar{z}$ we obtain

$$2GD = \eta\int_{z_0}^{z} P\,d\zeta + (3 - 4\nu)\varphi(z) - z\bar{\varphi}'(\bar{z}) - \bar{\psi}(\bar{z}) \tag{2.7}$$

We introduce the analytic function /10/

$$\Omega(z) = \bar{\Phi}(z) + z\Phi'(z) + \bar{\Psi}(z), \quad \Phi(z) = \varphi'(z), \quad \Psi(z) = \psi'(z)$$

which we use to find a representation for the stress tensor and displacements of the fluid-saturated porous medium in terms of two analytic functions from (2.5) and (2.7)

$$\sigma_{22} - i\sigma_{12} + Q = \Phi(z) + \Omega(\bar{z}) + (z - \bar{z})\bar{\Phi}'(\bar{z}) \tag{2.8}$$

$$2G\left(\frac{\partial}{\partial x_1}w_1 + i\frac{\partial}{\partial x_1}w_2\right) - Q = \varkappa\Phi(z) - \Omega(\bar{z}) - (z - \bar{z})\bar{\Phi}'(\bar{z})$$

$$Q = \eta P + \eta\int_{z_0}^{z}\frac{\partial}{\partial\bar{z}}P\,d\zeta, \quad \varkappa = 3 - 4\nu$$

Specifying the load on the upper and lower crack edges, we obtain a Dirichlet problem on the exterior of a slit for two analytic functions $\Phi(z), \Omega(z)$. Using the superposition principle, we represent the stress and displacement fields in the form of the sum of two fields, one of which corresponds to a continuous body subjected to loads applied within the body ($\sigma_0$ is a homogeneous compressive stress, and $P_\infty$ is the unperturbed interstitial fluid pressure), while the second is a body with a slit to whose surfaces loads are applied. The boundary conditions on the crack edges here have the form

$$\sigma_{22}^{\pm} = \sigma_0 - P_c(x_1), \quad \sigma_{12}^{\pm} = 0, \quad x_2 = 0 \pm 0 \tag{2.9}$$

Moreover, the values of the function $Q$ on the crack edges must be given to solve the boundary-value problem (2.8). It follows from (1.7) and (2.8) that

$$Q^{\pm} = Q_1^{\pm} + iQ_2^{\pm}$$

$$Q_1^{\pm} = \eta(P_c(x_1, t) - P_\infty) + {}^1/_2\eta[P_c(x_1, t) - P_c(x_0, t)]; \quad Q_2^{\pm} = \mp{}^1/_2\frac{\eta\mu}{k}\int_{x_0}^{x_1} v\,d\zeta$$

$$(x_0 = z_0, \operatorname{Im} z_0 = 0).$$

The solution of the Dirichlet boundary-value problem (2.8) and (2.9) is known /10/

$$\begin{Bmatrix}\Phi(z)\\ \Omega(z)\end{Bmatrix} = -\frac{1}{2\pi i\sqrt{z^2 - l^2}}\int_{-l(t)}^{l(t)}\frac{\sqrt{\tau^2 - l^2}\,R(\tau,t)}{\tau - z}\,d\tau \mp \frac{1}{2\pi}\int_{-l(t)}^{l(t)}\frac{T(\tau,t)}{\tau - z}\,d\tau + \frac{c_1}{\sqrt{z^2 - l^2}} \tag{2.10}$$

$$c_1 = -\frac{1}{2\pi}\frac{\varkappa - 1}{\varkappa + 1}\int_{-l(t)}^{l(t)} T(\tau, t)\,d\tau, \quad T(\tau, t) = \frac{1}{2}\frac{\eta\mu}{k}\int_{x_0}^{\tau} v(\zeta, t)\,d\zeta$$

$$R(x_1, t) = \sigma_0 - P_c(x_1, t) + \eta(P_c(x_1, t) - P_\infty) + {}^1/_2\eta(P_c(x_1, t) - P_c(x_0, t)) \tag{2.11}$$

$$\frac{\partial}{\partial x_1} w_2 = \frac{\varkappa+1}{4G}\left\{-T(x_1, t) + \frac{1}{\pi\sqrt{l^2 - x_1^2}} \int_{-l}^{l} \frac{\sqrt{l^2-\tau^2}\, R(\tau, t)}{\tau - x_1}\, d\tau - \frac{2c_1}{\sqrt{l^2 - x_1^2}}\right\}, \quad |x_1| \leqslant l, \quad x_2 = 0$$

The condition of symmetry of the fluid pressure in the crack ($P_c(-x_1, t) = P_c(x_1, t)$) was taken into account in finding the constant $c_1$.

Since the load is applied symmetrically to the crack edges, the crack profile is also symmetric with respect to the origin: $w_2(-x_1, t) = w_2(x_1, t)$. Consequently $\partial w_2(-x_1, t)/\partial x_1 = -\partial w_2(x_1, t)/\partial x_1$. The velocity of fluid leakage from the crack is also symmetrical about the origin: $v(-x_1, t) = v(x_1, t)$. To satisfy these symmetry conditions it is necessary to set $x_0 = 0$. We here obtain

$$T(x_1, t) = \frac{1}{2}\frac{\eta\mu}{k}\int_0^{x_1} v\, d\zeta, \quad c_1 = 0 \tag{2.12}$$

Using (2.12), we find from (2.11) after integration with respect to $x_1$ and simple reduction

$$w(x_1, t) = \frac{\varkappa+1}{4G}\left\{\frac{2}{\pi}\int_0^{l(t)}\left[P_c(\zeta, t) - \sigma_0 - \eta(P_c(\zeta, t) - P_\infty) - \frac{1}{2}\eta(P_c(\zeta, t) - P_c(0, t))\right]\ln\frac{\sqrt{l^2 - x_1^2} + \sqrt{l^2 - \zeta^2}}{\sqrt{|\zeta^2 - x_1^2|}}\, d\zeta - \frac{\eta\mu}{k}\left[(l - x_1)\int_0^{l} v(\zeta, t)\, d\zeta + \int_{x_1}^{l}(x_1 - \zeta)\, v(\zeta, t)\, d\zeta\right]\right\} \tag{2.13}$$

We obtain the stress intensity factor $K_{\rm I}$ from (2.8), (2.10) and (2.12) by taking account of the square root singularity of the stress tensor $\sigma_{22}$ becoming infinite near the ends of the symmetrically loaded crack

$$K_{\rm I} = 2\sqrt{\frac{l}{\pi}}\int_0^{l}\left[P_c(\zeta, t) - \sigma_0 - \eta(P_c(\zeta, t) - P_\infty) - \frac{1}{2}\eta(P_c(\zeta, t) - P_c(0, t))\right]\frac{d\zeta}{\sqrt{l^2 - \zeta^2}} \tag{2.14}$$

For $\eta = 0$ expressions (2.13) and (2.14) yield the solution of the classical problem of a quasistationary crack of normal separation in an elastic medium. Components which depend on the velocity of fluid leakage from the crack occur in the expression for the crack opening in consolidation theory.

The interstitial fluid pressure is described by a diffusion equation of the piezoconductivity type that can be obtained from (1.3) and (2.5)

$$\frac{\partial}{\partial t} P = c\Delta P - \omega\frac{\partial}{\partial t}(\operatorname{Re}\Phi) \tag{2.15}$$

$$c = \frac{(1+\nu)\, kGB}{(1-\nu)\,\eta\mu\,(3 - 4\eta B)}, \quad \omega = \frac{4(1+\nu)\, B}{3 - 4\eta B}$$

where $c$ is the consoliation or diffusion coefficient /3/.

The analytic function $\Phi(z)$, in (2.15) can be converted to the form

$$\Phi(z) = -\frac{z}{\pi i\sqrt{z^2 - l^2}}\int_0^{l}\frac{\sqrt{\zeta^2 - l^2}\, R(\zeta, t)}{\zeta^2 - z^2}\, d\zeta - \frac{\eta\mu}{4\pi k}\int_0^{l} v(\zeta, t)\ln\frac{l^2 - z^2}{\zeta^2 - z^2}\, d\zeta \tag{2.16}$$

Taking the real part of the complex function (2.16) and changing to the real variables $x_1, x_2$, we obtain

$$\operatorname{Re}\Phi(z) = -\frac{\eta\mu}{4\pi k}\int_0^l v(\zeta, t) K(\zeta, l; x_1, x_2)\, d\zeta + \tag{2.17}$$

$$\frac{1}{2\pi\sqrt{\rho_+\rho_-}}\int_{-l}^{l} R(\zeta, t)\frac{\sqrt{l^2-\zeta^2}\cos[\alpha(\zeta)+(\alpha(l)+\alpha(-l))/2]}{\sqrt{(x_1-\zeta)^2+x_2^2}}\, d\zeta$$

$$K(\zeta, l; x_1, x_2) = \frac{1}{2}\ln\frac{((l-x_1)^2+x_2^2)((l+x_1)^2+x_2^2)}{((\zeta-x_1)^2+x_2^2)((\zeta+x_1)^2+x_2^2)}$$

$$\rho_\pm = \sqrt{(x_1^2 \pm l)^2 + x_2^2}, \quad \alpha(y) = \arccos\frac{x_1 - y}{\sqrt{(x_1-y)^2+x_2^2}}$$

Therefore, a hydraulic fault in a porous fluid-saturated stratum is described by a system of non-linear integrodifferential equations including the equations of fluid motion in the crack (1.6), a piezoconductivity equation with a source (2.15) and (2.17), a functional relation between the crack opening and the pressure and the velocity of fluid leakage from the crack (2.13), and an expression for $K_I(t)$ (2.14). In the general case this system of equations can only be solved by numerical methods.

*3. A stationary crack;* $l = \text{const}$. The crack length does not change with time if the stress intensity factor is less than the adhesion modulus of the rock.

We consider the case when the fluid is injected into the crack at a constant pressure $P_0$. If the hydraulic conductivity of the crack is considerably greater than the conductivity of the medium, then (1.6) reduce to the boundary conditions on the crack contour

$$P(x_1, t) = P_0 \tag{3.1}$$

Using the Green's-function method /11/, we represent the solution of the piezoconductivity Eq.(2.15) in the form

$$P(x_1, x_2, t) = P_1(x_1, x_2, t) - \tag{3.2}$$

$$\omega\int_0^t d\tau\int_0^\infty d\xi\int_{-\infty}^{\infty} d\zeta\frac{\partial}{\partial t}(\operatorname{Re}\Phi) G(x_1-\zeta, x_2-\xi; t-\tau) + P_\infty$$

$$G(x_1-\zeta, x_2-\xi; t-\tau) = \frac{1}{4\pi c(t-\tau)}\left\{\exp\left[-\frac{(x_1-\zeta)^2+(x_2-\xi)^2}{4c(t-\tau)}\right] + \exp\left[-\frac{(x_1-\zeta)^2+(x_2+\xi)^2}{4c(t-\tau)}\right]\right\}$$

The pressure $P_1(x_1, x_2, t)$ is determined by the potential of a simple layer /11/

$$P_1(x_1, x_2, t) = \int_0^t \frac{d\tau}{4c\pi(t-\tau)}\int_{-\infty}^{\infty}\frac{c\mu}{k} v(\zeta, \tau)\exp\left[-\frac{(x_1-\zeta)^2+x_2^2}{4c(t-\tau)}\right] d\zeta \tag{3.3}$$

Substituting (3.3) into (3.2) and taking account of condition (3.1), we obtain an integral equation to determine the rate of fluid leakage from the crack $(x_2 = 0)$

$$\int_0^t \frac{d\tau}{4c\pi(t-\tau)}\int_{-l}^{l}\frac{c\mu}{k} v(\zeta, \tau)\exp\left[-\frac{(x_1-\zeta)^2}{4c(t-\tau)}\right] d\zeta - \tag{3.4}$$

$$\omega\int_0^t\int_0^\infty\int_{-\infty}^{\infty}\frac{\partial}{\partial t}(\operatorname{Re}\Phi) G(x_1-\zeta, -\xi; t-\tau)\, d\tau\, d\xi\, d\zeta = P_0 - P_\infty$$

Applying a Laplace transform to this equation and taking account of the expression for the source $\partial(\operatorname{Re}\Phi)/\partial t$ (2.17) we obtain after reduction ($K_0(z)$ is the Macdonald function)

$$\int_{-l}^{l} v(\xi, p) K_0(\sigma|x_1-\xi|)\, d\xi + \lambda\left\{\int_{-l}^{l} v(\xi, p) M(\sigma|x_1-\xi|)\, d\xi - \frac{1}{2}A(l, \sigma)\int_{-l}^{l} v(\xi, p)\, d\xi = \frac{\delta}{p}\right. \tag{3.5}$$

$$M(\sigma|x_1-\xi|) = K_0(\sigma|x_1-\xi|) + \ln(\sigma|x_1-\xi|)$$

$$A(l, \sigma) = K_0(\sigma|x_1-l|) + K_0(\sigma|x_1+l|) + \ln(\sigma^2|l^2-x_1^2|)$$

$$\sigma = \sqrt{p/c}, \quad \lambda = \omega\eta/8, \quad \delta = 2\pi k(P_0 - P_\infty)/\mu$$

The parameter $\lambda$ characterizes the effect of the reaction of the strain of the medium on the fluid filtration. For $\lambda = 0$ Eq.(3.5) corresponds to uncoupled consolidation.

Eq.(3.5) is among the class of convolution-type singular integral equations that has been studied in detail. It has been shown /12/ for equations with Macdonald zero-th order kernels that on decomposing the solution into Mathieu functions, (3.5) reduces to an infinite system of linear algebraic equations that can be approximated by a finite system of equations. By obtaining the solution of these and applying the inverse Laplace transform, the unknown rate of fluid leakage from the crack $v(x_1, p)$ $x_1 \in [-l, l]$ can be obtained with the necessary accuracy. Knowing the rate of fluid leakage, and the pressure in the crack, the crack opening, the stress intensity factor, and the interstitial fluid pressure distribution in the stratum can be obtained using (2.13), (2.14), (2.17), (3.2), and (3.3).

The integral Eq.(3.5) for the Laplace transform of the rate of fluid leakage from the crack $v(x_1, p)$ $x_1 \in [-l, l]$ provides the possibility of performing qualitative investigations of the problem under consideration, in particular, of finding the asymptotics forms of the solution for short times $(t \to 0)$ and long times $(t \to \infty)$.

*4. The asymptotic form of the solution of (3.5) for long times.* This case corresponds to small values of the parameter $p\,(p \to 0)$, i.e., values of the transforms for small $p$ yield the main contribution to the asymptotic form of the solution for long times on carrying out the inverse Laplace transformation.

Making use of the asymptotic formula for the Macdonald function for small arguments $(K_0(z) = \ln(z\gamma/2) + o(z)$, $\gamma = \exp\{c\}$; $c$ is Euler's constant), we convert (3.5) to the well-known Carleman equation /13/

$$\int_{-l}^{l} v(\xi, p) \ln |x_1 - \xi|^{-1} d\xi = \frac{\delta}{p} + N\left[\frac{1}{2}\ln\frac{p\gamma^2}{4c} + o(p)\right] \quad (4.1)$$

$$N = \int_{-l}^{l} v(\xi, p)\, d\xi$$

whose solution can be reduced to the form

$$v(x_1, p) = -\frac{N}{\pi^2\sqrt{l^2 - x_1^2}} \quad (4.2)$$

$$N = -\frac{2\delta}{p\ln(\alpha p) + o(p^2)}, \quad \alpha = \frac{1}{c}\left(\frac{l\gamma}{4}\right)^2$$

Applying the inverse Laplace transformation to (4.2) and evaluating the integral by the saddle-point method for long times $t\,(p \to 0)$ we obtain a formula for the rate of leakage of the fault fluid from the crack

$$v(x_1, t) = \frac{2\delta}{\pi^2\sqrt{l^2 - x_1^2}}\left[\frac{1}{\ln(t/\alpha)} + o\left(\frac{1}{\ln(t/\alpha)}\right)\right], \quad t \to \infty \quad (4.3)$$

that agrees with the corresponding solution of this same problem without taking account of the reaction of the medium strain on the filtration process, i.e., without taking account of the coherence effects.

Substituting (4.3) into (2.13), we obtain a formula for the crack opening

$$w(x_1, t) = \frac{\varkappa + 1}{4G}\left\{\left[P_0 - \eta(P - P_\infty) - \sigma_0\right]\sqrt{l^2 - x_1^2} - \right. \quad (4.4)$$

$$\left. \frac{\eta\mu\delta}{\pi^2 k}\left[\frac{\pi l}{2} - x_1 \arcsin\frac{x_1}{l} - \sqrt{l^2 - x_1^2}\right]\frac{1}{\ln(t/\alpha)} + o\left(\frac{1}{\ln(t/\alpha)}\right)\right\}$$

*5. The asymptotic of the solution of (3.5) for short times.* Large values of the parameter $p$ in the Laplace transform correspond to short times $(t \to 0)$. The approximate solution of the integral Eq.(3.5) can be obtained by the method of matched asymptotic expansions. We will find the degenerate solution of (3.5). Let $|x_1| \leqslant l - c_0 p^{-1/2+\varepsilon}$ $(c_0 \sim 1,\ 0 < \varepsilon < 1/2)$. Using the results from /14/ on the expansion of integrals with a delta-function type kernel in a parameter, the asymptotic representation of the Macdonald function for large values of the argument $(K_0(z) \sim \sqrt{\pi/(2z)}\exp\{-z\})$ and omitting components of order $p^{-1-\varepsilon}$ $(\varepsilon > 1/4)$, in (3.5), we also obtain a Carleman equation

$$\int_{-l}^{l} v(\xi, p) \ln \frac{1}{|x_1 - \xi|} d\xi = -\frac{\delta}{\lambda p} + \tag{5.1}$$
$$\frac{N}{2\lambda}[\ln|l^2 - x_1^2| + K_0(\sigma|l - x_1|) + K_0(\sigma|l + x_1|)]$$

The solution of (5.1) for large $p$ is given by the expression

$$v(\xi, p) = \frac{8\sqrt{l}\,\delta(\varkappa + 1)}{\pi c^{1/4}\lambda\Gamma^2(1/4)} \frac{1}{\sqrt{l^2 - x_1^2}} \left[\frac{1}{p^{3/4}} + o\left(\frac{1}{p^{3/4}}\right)\right] \tag{5.2}$$

Applying the inverse Laplace transformation to (5.2), we obtain the solution of the integral Eq.(3.5) for short times

$$v(\xi, t) = \frac{4\sqrt{2l}\,\delta(\varkappa + 1)}{\pi^2 c^{1/4}\lambda\Gamma(1/4)} \frac{1}{\sqrt{l^2 - x_1^2}} \left[\frac{1}{t^{1/4}} + o\left(\frac{1}{t^{1/4}}\right)\right] \tag{5.3}$$

In this case the crack opening has the form

$$w(x_1, t) = \frac{\varkappa + 1}{4G}\Big\{[P_0 - \sigma_0 - \eta(P_0 - P_\infty)]\sqrt{l^2 - x_1^2} - \tag{5.4}$$
$$\frac{8\sqrt{2l}\,\delta(1 + \varkappa)\eta\mu}{\pi^2 c^{1/4}\lambda\Gamma(1/4)k} \frac{1}{t^{1/4}}\left[\frac{\pi l}{2} - x_1 \arcsin\frac{x_1}{l} - \sqrt{l^2 - x_1^2} + o(1)\right]$$

Expressions (5.3) and (5.4) yield the solution of the problem of a fixed hydraulic fault crack in a medium being consolidated for short times but greater than a certain time $t_*$ $(t > t_*)$, where $t_*$ is the time necessary for the crack opening under jumplike loading of the slit by a constant pressure of the fault fluid when taking account of processes of swelling of the medium by the saturated fluid.

The flow of the fault fluid at $x_1 = 0$ (discharge) is given by the expression

$$Q = \frac{d}{dt}V_c + Q_c = h\left[\frac{d}{dt}\int_{-l}^{l} w(x_1, t)\,dx_1 + 2\int_{-l}^{l} v(x_1, t)\,dx_1\right] \tag{5.5}$$

where $h$ is the height of a vertical crack, $V_c$ is the crack volume, and $Q_c$ is the quantity of fluid filtering in the strata per unit time through the crack surface.

As $t \to 0$, by substituting expressions (4.3) and (4.4) into (5.5), we obtain

$$Q = \frac{8\sqrt{2l}\,\delta(1 + \varkappa)h}{\pi c^{1/4}\lambda\Gamma(1/4)}(t^{-1/4} + o(t^{-1/4})) \tag{5.6}$$

As $t \to \infty$ we find from (5.5), (3.3) and (3.4)

$$Q = \frac{4\delta h}{\pi}\left[\frac{1}{\ln(t/\alpha)} + o\left(\frac{1}{\ln(t/\alpha)}\right)\right] \tag{5.7}$$

The asymptotic forms obtained for the solution of the non-stationary problem of a hydraulic fault crack in a medium saturated by a fluid can be used to determine the stratum parameters by hydrodynamic methods /15/.

Keeping the pressure $P_0$ in the borehole constant, the stratum parameters, the permeability, porosity, etc., can be estimated on the basis of data on the change in the flow and (5.6) and (5.6).

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# ON THE RHEOLOGICAL INSTABILITY OF AN ELASTIC DAMAGING MEDIUM*

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A medium is examined that contains damage of the microcrack type scattered over the volume, whose number and dimensions can vary under the action of applied stresses. Such materials include brittle rocks, certain metal alloys, glass, etc. To describe the behaviour of such media a model of continuum fracture of elastic bodies /1/ is used based on the local balance between the effective surface energy of the microcracks and the cumulative elastic energy of the material surrounding the microcracks. The constraints on the allowable strain values imposed by the Hadamard condition /2/, which is a necessary condition for the correctness of any dynamic or quasistatic problems, are investigated in an isothermal approximation. These constraints play the part of a strength criterion that is closely associated with the internal structure of the rheological relationships used.

It is shown that violation of the Hadamard condition, identifiable with the rheological instability of the material, is accompanied by the formation of stationary surfaces of strain discontinuity, where, unlike an elastic-plastic dilating material /3, 4/, the origination of rheological instability is possible for the model being used in both the loading process and in unloading of the material. The orientation of the

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